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Hyperbolic space cosmologies

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Abstract: We present a systematic study of accelerating cosmologies obtained from M/string theory compactifications of hyperbolic spaces with time-varying volume. A set of vacuum solutions where the internal space is a product of hyperbolic manifolds is found to give qualitatively the same accelerating four-dimensional FLRW universe behavior as a single hyperbolic space. We also examine the possibility that our universe is a hyperbolic space and provide exact Milne type solutions, as well as intersecting S-brane solutions. When both the usual $4D$ spacetime and the m-dimensional internal space are hyperbolic, we find eternally accelerating cosmologies for $m \geq 7$, with and without form field backgrounds. In particular, the effective potential for a magnetic field background in the large 3 dimensions is positive definite with a local minimum and thus enhances the eternally accelerating expansion.

Keywords: M-Theory, Classical Theories of Gravity, Cosmology of Theories beyond the SM, Physics of the Early Universe.

Contents

1. Introduction

The past few years have produced a surge in the study of time-dependent solutions due to experimental data which point to cosmic acceleration in the present universe. While many effective inflationary scenarios may be devised to explain this phenomenon, one would hope that a natural candidate would emerge by studying a more fundamental theory such as superstring or M-theory. Until recently, however, it was believed that the low energy supergravity limit of string theory could not generate four dimensional acceleration from compactification. This argument [1], often regarded as a no-go theorem, is valid when the compact internal space is a time independent, non-singular compact manifold without boundary.

By explicit construction, Townsend and Wolhfarth showed in ref. [2] that vacuum solutions can produce accelerating cosmologies if one uses compact time dependent hyperbolic internal spaces. Their model was recognized [3] as the zero flux limit of the S2-brane solutions [4, 5], and the cosmology of these generalized S-brane solutions has also been studied. Further generalizations of these models were discussed in refs. [6]–[12]. Related cosmologies have been discussed in refs. [13]–[19]. The use of hyperbolic space for internal space was proposed in refs. [20, 21].

This class of S-brane solutions provides a way to obtain cosmic acceleration from time dependent compactification where the internal dimensions are hyperbolic and the usual $(3+1)$ dimensions are flat. In this and related scenarios, the typical amount of inflation is on the order of one, which is too small to solve the horizon and flatness problems in the early universe [7]–[10]. Consequently it is of utmost importance to try to find a suitable mechanism to generate a larger inflationary period.

One might suspect that since the effective potentials due to compactifications are always exponentials with coefficients of order one, the e-folding number within the acceleration phase should also be of order one. But it was pointed out in ref. [10] that an exponential potential with suitable coefficients of scalar fields can in principle produce eternal inflation. In this paper we make a further step by presenting solutions of eternally accelerating expansion for pure gravity compactified on hyperbolic spaces.

In our search for more general models with sufficient inflation, we have tried to make a systematic study of the models compactified on a product of hyperbolic and flat spaces. For general product space compactifications, solutions are generally difficult to obtain. Using a specific ansatz we find a new class of vacuum spacetimes which are a product of flat and hyperbolic spaces. We examine the possibility of using these spaces to obtain inflation but find that they do not lead to accelerating cosmologies when the external dimensions are flat.

We next turn to the prospect of having hyperbolic external dimensions. We first study the case where the external space is hyperbolic and the internal space is flat. As a further step we study the case where both the external and internal dimensions are hyperbolic since these spaces seem to be among the most promising for obtaining large inflation. In this case we find solutions whose late time behavior is approximately characterized as a Milne spacetime with nearly constant expansion. By studying perturbative expansions about such

Milne solutions, we find that it is possible to obtain eternally accelerating expansion when the dimension of the internal space is greater than or equal to seven, which is interesting in view of the possibility of embedding these solutions in M-theory. While it is not clear if this model is phenomenologically viable, at least we find that it is possible to improve the amount of inflation for M-theory compactifications.

This paper is organized as follows: In section 2, we discuss vacuum solutions to Einstein equations which are products of flat and hyperbolic spaces and show that they lead to a phase of accelerated expansion. Here we also review and further discuss why hyperbolic internal spaces help inflation in general. In section 3, we discuss the differences and advantages in treating hyperbolic spaces instead of flat spaces as the external space. In addition we find Milne type solutions as exactly solvable examples. In section 4, we discuss the product spaces by using the effective $4D$ action with scalar fields which parametrize the radii of the internal spaces. This viewpoint is useful because we can use our knowledge of scalar field inflation to increase our physical intuition of our results. The effective action approach is used in section 5 to analyze Milne type exact solutions for the case in which both the internal and external spaces are hyperbolic. Perturbations around Milne spacetimes are shown to lead to eternally accelerating expansion when the dimension of the internal space is greater than or equal to seven. Our result suggests that there is the possibility of obtaining big expansion from M-theory compactifications.

2. Product spaces: gravity viewpoint

2.1 Vacuum solutions for higher-dimensional gravity

In this section, we discuss and construct vacuum solutions of the Einstein equations. We focus primarily on solutions which can be written as a product of hyperbolic and flat spaces. In the case of these vacuum solutions, accelerated expansion is possible only if the internal dimensions include hyperbolic spaces, i.e. have negative curvature.

Let us consider a general spacetime with the following product space metric ansatz

$$ds_D^2 = -\mathrm{e}^{2A(t)}dt^2 + \sum_{i=0}^{n} \mathrm{e}^{2B_i(t)} d\Sigma_{m_i,\epsilon_i}^2 \,, \tag{2.1}$$

where the Σ_{m_i,ϵ_i} are m_i dimensional spaces with curvature specified by ϵ_i; the values of $\epsilon_i = 0, +1, -1$ correspond to the flat, spherical or hyperbolic spaces, respectively. The metric for each Σ is

$$d\Sigma_{m_i,\epsilon_i}^2 = \bar{g}_{ab}^{(i)} dz^a dz^b = \begin{cases} d\psi^2 + \sinh^2\psi \, d\Omega_{m_i-1}^2 \,, & \epsilon_i = -1 \,, \\ d\psi^2 + \psi^2 \, d\Omega_{m_i-1}^2 \,, & \epsilon_i = 0 \,, \\ d\psi^2 + \sin^2\psi \, d\Omega_{m_i-1}^2 \,, & \epsilon_i = +1 \,, \end{cases} \tag{2.2}$$

which have curvature

$$\bar{R}_{ab}^{(i)} = \epsilon_i (m_i - 1) \bar{g}_{ab}^{(i)} \,. \tag{2.3}$$

We will also denote the metrics in eq. (2.2) for m-dimensional flat, spherical and hyperbolic spaces by ds^2_{Rm}, ds^2_{Sm} and ds^2_{Hm}. The Ricci tensor for the metric (2.1) is relatively simple

$$R_{tt} = -\sum_{i=0}^{n} m_i(\ddot{B}_i + \dot{B}_i^2 - \dot{A}\dot{B}_i), \tag{2.4}$$

$$R_{ab}^{(i)} = \left\{ e^{2B_i - 2A} \left[\ddot{B}_i + \dot{B}_i \left(-\dot{A} + \sum_{j=0}^{n} m_j \dot{B}_j \right) \right] + \epsilon_i (m_i - 1) \right\} \bar{g}_{ab}^{(i)} . \tag{2.5}$$

To simplify the equations we use the gauge condition

$$-A + \sum_{j=0}^{n} m_j B_j = 0 . \tag{2.6}$$

The vacuum Einstein equations then become

$$-\sum_{i=0}^{n} m_i \ddot{B}_i + \sum_{i=0}^{n} m_i(m_i - 1)\dot{B}_i^2 + \sum_{j \neq i}^{n} m_i m_j \dot{B}_i \dot{B}_j = 0 , \tag{2.7}$$

$$\ddot{B}_i + \epsilon_i(m_i - 1) e^{2(m_i - 1)B_i} e^{2\sum_{j \neq i} m_j B_j} = 0 . \tag{2.8}$$

For spacetimes given in eq. (2.1), the D-dimensional Einstein frame metric is defined as

$$ds^2_D = \sum_{i=1}^{n} e^{2B_i(t)} d\Sigma^2_{m_i, \epsilon_i} + e^{-\frac{2}{d-1}\sum_{i=1}^{n} m_i B_i(t)} ds^2_{E,d+1} , \tag{2.9}$$

where $d \equiv m_0$, and the $(d+1)$-dimensional part of the metric is given by

$$ds^2_{E,d+1} = e^{\frac{2}{d-1}\sum_{i=1}^{n} m_i B_i} \left(-e^{2\sum_{j=0}^{n} m_j B_j} dt^2 + e^{2B_0} ds^2_d \right) . \tag{2.10}$$

2.2 Product space of same type subspaces

Due to the coupling inherent in the equations of motion for spaces with non-zero curvature, it is not known how to generally solve the coupled differential equations in eqs. (2.7) and (2.8). We can find exact solutions only in some relatively simple cases if we take a particular ansatz for the solution. As an interesting case that can be solved exactly, we consider the spacetime with a d-dimensional flat subspace, so $\epsilon_0 = 0$. For simplicity we assume that the rest of space is a product of spaces with the same curvature $\epsilon_1 = \cdots = \epsilon_n = \epsilon$. These spaces are of the form $\mathbb{R}^{1+d} \times \mathbb{M}_{m_1,\epsilon} \times \cdots \times \mathbb{M}_{m_n,\epsilon}$.

In this case, the equation for B_0 is easily solved

$$B_0 = \lambda_0 t + \lambda_1 , \tag{2.11}$$

with two integration constants λ_0 and λ_1. The constant λ_1 may be eliminated by a shift of the time in $\mathbb{R}^{1,d}$, and in the following discussion we take $\lambda_1 = 0$.

For the functions B_i we make the ansatz

$$B_i = -\frac{d\lambda_0}{m-1} t + \frac{\beta_i}{m-1} - \frac{1}{m-1} f(t) , \tag{2.12}$$

where we have defined $m = \sum_{i=1}^{n} m_i$ and

$$f(t) = \begin{cases} \ln\left(\sinh[\lambda(t-t_1)]\right), & \epsilon = -1, \\ \lambda(t-t_1), & \epsilon = 0, \\ \ln\left(\cosh[\lambda(t-t_1)]\right), & \epsilon = +1, \end{cases} \tag{2.13}$$

and β_i and λ are undetermined constants. The variable A in this case is then determined by the gauge condition (2.6), namely,

$$A = -\frac{d\lambda_0}{m-1}t + \frac{\beta}{m-1} - \frac{m}{m-1}f(t), \tag{2.14}$$

with $\beta = \sum_{i=1}^{n} m_i \beta_i$. For different values of i, the equations in eq. (2.8) determine the β parameters

$$\begin{aligned} \beta_i &= \frac{1}{2}\ln\left[\frac{\lambda^2}{(m-1)(m_i-1)}\prod_{j=1}^{n}\left(\frac{m_i-1}{m_j-1}\right)^{m_j}\right], \\ \beta &= \frac{m}{2}\ln\left[\frac{\lambda^2}{m-1}\prod_{i=1}^{n}(m_i-1)^{-m_i/m}\right], \end{aligned} \tag{2.15}$$

while λ is determined by eq. (2.7) to be

$$\lambda = \sqrt{\frac{d(m+d-1)}{m}}\,\lambda_0 . \tag{2.16}$$

These solutions are slight generalizations of those discussed in ref. [10] where product spaces of identical subspaces were considered with $m_i = m_1$ for all i. In this more generic case, the $(d+1)$-dimensional metric in the Einstein frame takes the form

$$ds_{d+1}^2 = -S^{2d}(t)dt^2 + S^2(t)ds_{Rd}^2, \tag{2.17}$$

with the scale factor

$$S(t) = \exp\left[\frac{-(m+d-1)\lambda_0 t + \beta - mf(t)}{(d-1)(m-1)}\right]. \tag{2.18}$$

In terms of the proper time defined by

$$d\tau = S^d(t)dt, \tag{2.19}$$

the metric (2.17) takes the standard FLRW form.

The conditions for expansion and accelerated expansion are, respectively,

$$\frac{dS}{d\tau} > 0, \qquad \frac{d^2S}{d\tau^2} > 0. \tag{2.20}$$

For the above solution, we find that the accelerated expansion is possible only for hyperbolic internal spaces and the conditions (2.20) are

$$\begin{aligned} n(t) \equiv -\sqrt{md}\coth\lambda(t-t_1) - \sqrt{m+d-1} > 0, \\ \frac{(m-1)d}{\sinh^2\lambda(t-t_1)} - n^2(t) > 0. \end{aligned} \tag{2.21}$$

JHEP10(2003)058

These conditions, which depend only on d and the total dimension of the internal space, are basically the same conditions that one obtains from a single internal space, and the expansion factor (the ratio of the scale factors at the starting and ending times of the accelerated expansion) is of order one [2] and [7]–[10].

Intuitively, the reason why we can get accelerated expansion only for hyperbolic space is that the hyperbolic internal spaces act as positive potentials in the dimensionally reduced effective $(d+1)$-dimensional theory. We will further discuss this effective potential viewpoint later.

2.3 The $\mathbb{M}_0 \times \mathbb{M}_1 \times \mathbb{M}_2$ spaces

Our product space compactifications to an Einstein manifold unfortunately do not give sufficient inflation as their behavior is similar to the original model. In the following discussions we will explore further possible inflationary mechanisms.

Consider a product of three spaces. Then the three coupled differential equations following from (2.8) must satisfy

$$B_0 = -\frac{(m_2-1)B_2 + m_1 B_1}{m_0} + \frac{1}{2m_0}\ln\left(\frac{-\ddot{B}_2}{\epsilon_2(m_2-1)}\right), \tag{2.22}$$

$$\epsilon_2(m_2-1)\,\ddot{B}_1\,\mathrm{e}^{2B_1} = \epsilon_1(m_1-1)\,\ddot{B}_2\,\mathrm{e}^{2B_2}\,, \tag{2.23}$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\left(\ln\left(-\epsilon_2\ddot{B}_2\right)\right) &= 2m_1\ddot{B}_1 + 2(m_2-1)\ddot{B}_2 - 2\epsilon_0 m_0(m_0-1)\,\mathrm{e}^{\frac{(m_0-1)}{m_0}\ln\left(\frac{-\ddot{B}_2}{\epsilon_2(m_2-1)}\right)} \times \\ &\quad \times \mathrm{e}^{\frac{2}{m_0}(m_0+m_2-1)B_2+m_1B_1)}.\end{aligned} \tag{2.24}$$

It is difficult to find a solution when $\epsilon_0 \neq 0$ and all the internal spaces are non-flat. However, in some cases such as $\mathbb{R}^1 \times \mathbb{H}_3 \times \mathbb{H}_m$, we can obtain exact solutions with $H \propto 1/t$. This will be discussed in subsections 3.3 and 5. Here we consider the solvable case with $\epsilon_0 = 0$ and take $m_0 = d = 3$, but we will put no restriction on the number of internal dimensions m_1, m_2.

2.4 $\mathbb{R}^{3+1} \times \mathbb{R}^{m_1} \times \mathbb{H}_{m_2}$

One of the simplest examples we can solve is the case where the internal space is the product of flat and hyperbolic spaces. We define

$$B_0(t) = \lambda_0 t\,, \qquad B_1(t) = a(t) - \frac{3\lambda_0 t}{m_1+m_2-1}\,, \qquad B_2(t) = b(t) - \frac{3\lambda_0 t}{m_1+m_2-1}\,, \tag{2.25}$$

so that the vacuum Einstein equations (2.22), (2.23) and (2.7) further simplify to

$$\begin{aligned}&\ddot{b} = (m_2-1)\,\mathrm{e}^{2m_1a(t)+2(m_2-1)b(t)}\,, \qquad \ddot{a} = 0\,,\\ &m_1(m_1-1)\dot{a}^2 + m_2(m_2-1)\dot{b}^2 - m_2\ddot{b} + 2m_1m_2\dot{a}\dot{b} = \frac{3(m_1+m_2+2)}{(m_1+m_2-1)}\,\lambda_0^2\,,\end{aligned} \tag{2.26}$$

while eq. (2.24) gives only a consistency condition. The set of the above equations has the solution

$$a(t) = \alpha_0 t\,, \qquad b(t) = -\frac{m_1}{m_2-1}\,\alpha_0 t + \frac{1}{m_2-1}\ln\left(\frac{\beta}{\sinh((m_2-1)\beta\, t)}\right), \tag{2.27}$$

with

$$\beta^2 = \frac{m_1(m_1+m_2-1)\alpha_0^2}{m_2(m_2-1)^2} + \frac{3(m_1+m_2+2)}{m_2(m_2-1)(m_1+m_2-1)}\,\lambda_0^2\,. \tag{2.28}$$

For the case of our interest, $m_1 = 1, m_2 = 6$, the metric in Einstein frame takes the form

$$ds^2 = \mathrm{e}^{\frac{7}{2}\lambda_0 t - a(t) - 6b(t)} ds_{E4}^2 + \mathrm{e}^{-\lambda_0 t + 2a(t)}\, dr^2 + \mathrm{e}^{-\lambda_0 t + 2b(t)} ds_{H6}^2\,, \tag{2.29}$$

where ds_E^2 is given by (2.17) with $d = 3$ and the scale factor is

$$S(t) = \mathrm{e}^{-\frac{3\lambda_0 t}{4} + \frac{a(t)}{2} + 3b(t)}\,. \tag{2.30}$$

If we define the four-dimensional proper time τ via

$$d\tau = \pm\, S^3(t) dt\,, \tag{2.31}$$

then the $4D$ spacetime is expanding if $dS/d\tau > 0$, namely, if $n(t) < 0$ or $n(t) > 0$, for the plus or minus sign in (2.31), where

$$n(t) = \frac{\alpha_0}{10} + 3\beta\,\coth(5\beta\, t) + \frac{3\lambda_0}{4}\,. \tag{2.32}$$

Let us write

$$\alpha_0 = c\,\lambda_0\,, \qquad \text{so} \qquad \beta = \frac{|\lambda_0|}{10}\sqrt{4c^2+15}\,,$$

where c is some number. Then the condition for acceleration $d^2S/d\tau^2 > 0$ is

$$\frac{15}{2}\frac{\beta^2}{\sinh^2(5\beta\, t)} > \left(\frac{(2c+15)\lambda_0}{20} + 3\beta\coth\left(5\beta\, t\right)\right)^2\,. \tag{2.33}$$

This condition is satisfied for both the positive and negative time interval, by suitably choosing c. An interesting case is $c = 1/2$, so $\beta = 4|\alpha_0|/5$. In this case, the time-varying volume factor $e^{2B_1(t)}$ of the $\mathbb{R}^1$ becomes unity. The condition for acceleration (as well as the condition for expansion) is satisfied in the interval $t_1 > t > t_2$ (or $t_1 < t < t_2$ depending on the choice $\lambda_0 < 0$ or $\lambda_0 > 0$), where

$$t_1 = \frac{1}{8\alpha_0}\ln\left(\frac{2-\sqrt{3}}{5}\right), \qquad t_2 = \frac{1}{8\alpha_0}\ln\left(\frac{2+\sqrt{3}}{5}\right). \tag{2.34}$$

That is, the $4D$ spacetime is accelerating in the interval $1.4631 > 4|\alpha_0|t > 0.1462$; during this interval the universe expands by the factor of

$$\frac{S(\tau_2)}{S(\tau_1)} = 3.3810\,. \tag{2.35}$$

This is a small improvement over the decomposition $\mathbb{R}^{3+1}\times\mathbb{H}_7$. It may be slightly further enhanced for smaller values of c.

2.5 $\mathbb{R}^{3+1} \times \mathbb{H}_{m_1} \times \mathbb{H}_{m_2}$ including radii

In the previous sections, we have normalized ϵ_i to $+1$, 0 or -1. Here we discuss the effect of including the radii factors r_i into the metric

$$ds_m^2 = r_1^2 ds_{Hm_1}^2 + r_2^2 ds_{Hm_2}^2 \,, \tag{2.36}$$

where r_1 and r_2 are the physical curvature radii of $\mathbb{H}_{m_1}$ and $\mathbb{H}_{m_2}$. Then the following $(4+m)$-dimensional metric ansatz parametrized by the function $K = K(t)$:

$$ds_{4+m}^2 = \mathrm{e}^{2\lambda_0 t}\, ds_{R3}^2 + \mathrm{e}^{-\frac{6\lambda_0 t}{m-1}} \left(- K^{\frac{2m}{m-1}}\, dt^2 + K^{\frac{2}{m-1}}\, ds_m^2 \right) , \tag{2.37}$$

solves the vacuum Einstein equations when

$$K(t) = \frac{\lambda_0 r_1 \gamma}{\sinh\left(\lambda_0 \beta |t - t_1|\right)} , \tag{2.38}$$

where

$$\beta = \sqrt{\frac{3(m+2)}{m}} \,, \qquad \gamma = \sqrt{\frac{3(m+2)}{m(m-1)(m_1-1)}} \,, \qquad r_1 = r_2 \sqrt{\frac{m_1 - 1}{m_2 - 1}} \,. \tag{2.39}$$

The 4-dimensional Einstein metric is read off as (2.17) with the scale factor

$$S(t) = e^{-\frac{3}{4}\lambda_0 t} K^{\frac{7}{12}} \,, \tag{2.40}$$

for $m = 7$. This is precisely the same scale factor one obtains for a single hyperbolic space studied in [2], and there is not much effect of introducing radii. We find that accelerated expansion is again possible but it does not give enough expansion factor.

3. Hyperbolic external space

In previous analysis of the class of time-dependent S-brane solutions, the external space was taken to be flat. In such cases there was a period of accelerated expansion but the late time limit of these solutions was decelerating. In this section, we consider the consequences of using a hyperbolic space instead of a flat space as the large spatial dimensions. We begin with a review of the Milne solutions and vacuum solutions with hyperbolic internal space. Then we re-examine the vacuum solutions from section 2, the difference being that we now go into Einstein frame for the hyperbolic space. In this case we find that the period of accelerated expansion vanishes but the late time behavior has nearly constant expansion characterized by Milne spacetimes. Milne spacetimes will later play a key role in our search for solutions with eternally accelerating expansion.

3.1 Milne spacetime limit

Let us first consider the particular limit $\lambda_0 \to 0$. Nontrivial solutions are obtained in this limit only for the hyperbolic product space. (The topology of the whole spacetime is

$\mathbb{R}^{d+1} \times \mathbb{H}_{m_1} \times \cdots \times \mathbb{H}_{m_n}$.) Taking this limit in eq. (2.12), we have $B_0 = 0$ and

$$\begin{aligned} B_i &= \frac{\bar\beta_i - \ln(t-t_1)}{m-1}\,, \\ A &= \frac{\bar\beta - m\ln(t-t_1)}{m-1}\,, \end{aligned} \tag{3.1}$$

where $m = \sum_{i=1}^n m_i$ and

$$\begin{aligned} \bar\beta_i &= \frac{1}{2}\ln\left[\frac{1}{(m-1)(m_i-1)}\prod_{j=1}^n\left(\frac{m_i-1}{m_j-1}\right)^{m_j}\right], \\ \bar\beta &= \frac{m}{2}\ln\left[\frac{1}{m-1}\prod_{i=1}^n (m_i-1)^{-m_i/m}\right]. \end{aligned} \tag{3.2}$$

Therefore, the higher-dimensional vacuum solution, for $t_1 = 0$, is

$$ds^2 = -\mathrm{e}^{\frac{2\bar\beta}{m-1}}\, t^{-\frac{2m}{m-1}} dt^2 + ds^2_{Rd} + t^{-\frac{2}{m-1}}\sum_{i=1}^n \mathrm{e}^{\frac{2\bar\beta_i}{m-1}} ds^2_{Hm_i}\,. \tag{3.3}$$

This is a generalization of Milne metric. For example when $n = 1$, the metric becomes

$$ds^2 = -d\xi^2 + ds^2_{Rd} + \xi^2 ds^2_{Hm}\,, \tag{3.4}$$

where $\xi = [(m-1)t]^{-1/(m-1)}$. Ref. [18] discusses some simple cosmological string models from Milne spacetime in four dimensions, i.e., with $d = 1$ and $m = 2$.

We remind the reader that if we do not take quotients of the Milne space, the above Milne spacetime can be obtained from a Wick rotation of Euclidean space so these spacetimes are flat with all Riemann curvature components vanishing. In fact these solutions cover just the patch of Minkowski space existing between the lightcone of an observer. The hyperbolic space in this case is just a result of a particular foliation of the spatial slices of flat space. The maximal extension of a Milne universe is flat Minkowski space which is not expanding. To understand this as an expanding spacetime we must take a quotient of the hyperbolic space as we discussed earlier. (However since this quotient does not affect the equations of motion, we leave the quotient implicit here.)

3.1.1 Flat space with hyperbolic extra dimensions

Before examining the hyperbolic external space, let us consider the hyperbolic internal space. We then find that the Einstein-frame $(d+1)$-dimensional metric obtained from (3.3) has the form of (2.17) with

$$S(t) = \exp\left[\frac{\bar\beta - m\ln(t-t_1)}{(m-1)(d-1)}\right] = \exp\left[\frac{\bar\beta}{(m-1)(d-1)}\right]\,(t-t_1)^{-\frac{m}{(d-1)(m-1)}}\,. \tag{3.5}$$

We may redefine the time coordinate as $(t_1 = 0)$

$$d\tau = -S^d(t)dt\,. \tag{3.6}$$

We then have

$$\tau = \frac{(d-1)(m-1)}{m+d-1} \exp\left[\frac{d\bar{\beta}}{(d-1)(m-1)}\right] t^{-\frac{m+d-1}{(d-1)(m-1)}}, \tag{3.7}$$

so that the metric can be written in the standard FLRW form as

$$ds^2_{E(d+1)} = -d\tau^2 + S^2(\tau) ds^2_{Rd}, \tag{3.8}$$

with

$$S(\tau) = \exp\left(-\frac{\bar{\beta}}{m-d-1}\right)\left[\frac{m+d-1}{(d-1)(m-1)}\tau\right]^{\frac{m}{m+d-1}}. \tag{3.9}$$

The Hubble parameter is

$$H = \frac{\partial_\tau S}{S} = \frac{m}{m+d-1}\frac{1}{\tau}. \tag{3.10}$$

Given that $d > 1$, the value $m/(m+d-1) < 1$ and hence $\partial_\tau^2 S < 0$. Note that this result, dependent only on the total dimension m of internal spaces, is valid also for Milne spacetime. We conclude that there is no inflation for the above generalized Milne solutions (3.3) either. Next we turn to the case where the usual $4D$ spacetime (i.e., the external space) is hyperbolic.

3.1.2 Interchanging external and internal spaces

If we identify the large spatial d dimensions as the hyperbolic part of the spacetime given by (3.3), with the labels d and m interchanged, the induced metric on $\mathbb{R}\times\mathbb{H}_d$ in the Einstein frame is given as (2.17) with the scale factor

$$S(t) = \exp\left[\frac{\bar{\beta}-\bar{\beta}_1}{(d-1)^2}\right] t^{-\frac{1}{d-1}}. \tag{3.11}$$

The internal space is then a flat m-dimensional space. As its size is fixed, the internal space can be completely ignored, and its dimension m is irrelevant to our solution.

We may redefine the time coordinate as in (3.6) and then get

$$\tau = (d-1)\exp\left[\frac{d(\bar{\beta}-\bar{\beta}_1)}{(d-1)^2}\right] t^{-\frac{1}{d-1}}, \tag{3.12}$$

so that the metric can be written in the standard FLRW form as

$$ds^2_{d+1} = -d\tau^2 + S^2(\tau)\, ds^2_{H_d}, \tag{3.13}$$

with

$$S(\tau) = \exp\left(-\frac{\bar{\beta}-\bar{\beta}_1}{d-1}\right)\left(\frac{\tau}{d-1}\right). \tag{3.14}$$

These Milne spacetimes have linear expansion, and it is also clear that the dimensional reduction of flat directions does not affect this result. The Hubble parameter is

$$H = \frac{\partial_\tau S}{S} = \frac{1}{\tau}. \tag{3.15}$$

This is the critical case with zero acceleration $\partial_\tau^2 S = 0$, and this fact will be used later.

JHEP10(2003)058

Since hyperbolic internal spaces turn on positive effective potentials in the effective $(d+1)$-dimensional theory, it is natural to expect that if we turn the internal flat space into hyperbolic spaces, we may increase the amount of inflation generated. We study this possibility in more detail in Subsec. 3.3.

3.2 Hyperbolic space with flat extra dimensions

Our exact solutions in subsection 2.1 can be readily reinterpreted as solutions on hyperbolic (or spherical or flat) external spaces with flat extra dimensions. Using (2.12), (2.14) and (2.16), and renaming some variables, we rewrite the spacetime metric (2.1) as

$$\begin{aligned} ds^2 &= -\mathrm{e}^{2dg(t)-\frac{2m}{d-1}(\lambda_0 t+\lambda_1)}dt^2 + \mathrm{e}^{2g(t)-\frac{2m}{d-1}(\lambda_0 t+\lambda_1)}d\Sigma_{d,\epsilon}^2 + \mathrm{e}^{2(\lambda_0 t+\lambda_1)}ds_{Rm}^2 \\ &= \mathrm{e}^{-\frac{2m}{d-1}(\lambda_0 t+\lambda_1)}ds_E^2 + \mathrm{e}^{2(\lambda_0 t+\lambda_1)}ds_{Rm}^2 \,, \end{aligned} \tag{3.16}$$

where the $(d+1)$-dimensional Einstein frame metric for the external space is given by

$$ds_E^2 = -\mathrm{e}^{2dg(t)}dt^2 + \mathrm{e}^{2g(t)}d\Sigma_{d,\epsilon}^2. \tag{3.17}$$

Here d is the dimension of the external space, which is flat, spherical or hyperbolic for $\epsilon = 0, 1, -1$, respectively. The function $g(t)$ is given by

$$g(t) = \begin{cases} \dfrac{1}{d-1}\ln\dfrac{\beta}{\sinh[(d-1)\beta(t-t_1)]} & : \epsilon = -1\,, \\ \pm\beta(t-t_1) & : \epsilon = 0\,, \\ \dfrac{1}{d-1}\ln\dfrac{\beta}{\cosh[(d-1)\beta(t-t_1)]} & : \epsilon = +1\,, \end{cases} \tag{3.18}$$

$$\beta = \frac{1}{d-1}\sqrt{\frac{m(m+d-1)}{d}}\,\lambda_0\,. \tag{3.19}$$

One can also derive the metric (3.4) from (3.16) by taking the limit $\lambda_0 \to 0$. This is the exact solution with a critical expansion (zero acceleration) discussed in the previous subsection. Here we examine the solutions before discussing a limiting behavior, in order to check whether we may get eternal or larger expansion.

The scale factor for the general exact solution (3.17) is simply $S(t) = \mathrm{e}^{g(t)}$ with $g(t)$ given in (3.18). The conditions for expansion and accelerated expansion are

$$0 < \frac{dS}{d\tau} = -\frac{\beta}{S^{d-1}}\coth(d-1)\beta t \tag{3.20}$$

$$0 < \frac{d^2S}{d\tau^2} = -\frac{(d-1)\beta^2}{S^{2d-1}}\,. \tag{3.21}$$

Obviously the first condition can be satisfied for $t < 0$, but the second condition never holds. There is no acceleration for this exact solution. However, we can see that the Milne spacetime limit $\beta \to 0$ reproduces the critical expansion.

JHEP10(2003)058

3.3 Hyperbolic space with flat and hyperbolic extra dimensions

In the previous subsection we examined the amount of inflation we could obtain when the internal dimensions were flat and the external dimensions were hyperbolic. Since hyperbolic spaces tend to improve inflation, we here turn to the case where both the internal and external spaces have hyperbolic dimensions. The solutions obtained in subsection 2.1 for the product spaces $\mathbb{R} \times \mathbb{H}_{m_1} \times \mathbb{R}_d \times \mathbb{H}_{m_2}$ are summarized in eqs. (2.12), (2.14) and (2.1). Instead of treating $\mathbb{R}_d$ as the external space, we now treat $\mathbb{H}_{m_1}$ as the external space with d and m_1 exchanged, and the remaining flat and hyperbolic with the example in the previous section, we have an additional hyperbolic space $\mathbb{H}_{m_2}$ for the internal space. In this case, the Einstein metric for the external space is

$$ds^2_{E,d+1} = \mathrm{e}^{\frac{2}{d-1}(dB_0+m_2B_2)} \left(-e^{2A}dt^2 + e^{2B_1} ds^2_{Hd}\right), \tag{3.22}$$

where the scale factor

$$S = \mathrm{e}^{\frac{A-B_1}{d-1}} = \sinh^{\frac{-1}{d-1}} \lambda(t-t_1)\ \mathrm{e}^{\frac{\beta-\beta_1}{(d-1)(m-1)}} \tag{3.23}$$

can be simply written as

$$S = a_0 \sinh^x \lambda(t-t_1)\,, \tag{3.24}$$

where the constants are $x = -1/(d-1)$ and $a_0 = \mathrm{e}^{\frac{\beta-\beta_1}{(d-1)(m-1)}}$. The key difference in taking the external space to be hyperbolic instead of flat shows up in the expansion factor S. In this case the expansion factor is a power of hyperbolic sine while in the other case the expansion factor is a power of hyperbolic sine times an exponential function. We will find a significant difference in the behavior of the universe.

Let us first examine the late time ($t \simeq 0_-$) and early time ($t \to -\infty$) behavior of the scale factor. The late time asymptotic behavior tells us whether there is eternally accelerating expansion. To leading order, the scale factor close to $t=0$ is

$$S \sim a_0(\lambda t)^x\,, \tag{3.25}$$

where we have set $t_1 = 0$. Surprisingly, different values of t_1 does not seem to significantly change the results. The proper time is defined as $d\tau = -S^d dt \sim -a_0^d(\lambda t)^{xd}dt$, so the relationship between the time t and the proper time τ is given by

$$\tau \sim -\frac{a_0^d(\lambda t)^{xd+1}}{xd+1} \sim (d-1)a_0^d(\lambda t)^{xd+1} \quad \Rightarrow \quad t \sim \tau^{-(d-1)}\,. \tag{3.26}$$

Writing the scale factor in terms of the proper time we find

$$S \sim a_0\lambda^x\,\tau^{-(d-1)x} \sim a_0\lambda^x\,\tau\,. \tag{3.27}$$

This solution has constant expansion but no eternally accelerating expansion. It is a Milne type solution so it is closer to an inflationary solution than when the external space was flat.

It turns out that the scale factor coincides with that in the previous section with $\lambda = (d-1)\beta$, and thus we see from (3.20) and (3.21) that the exact solution is unfortunately always decelerating. Using the above exact solution, we find that in the $t \to 0$ limit and

$d = 3$, the deceleration to first order scales as

$$\frac{d^2 S}{d\tau^2} \approx -\tau^{-5} \,. \tag{3.28}$$

In the other limit $t \to -\infty$, the function sinh can be approximated by the exponential function. It is easy to see in this case that

$$S \propto \tau^{1/3} \,, \tag{3.29}$$

which is a dust filled universe not in the phase of inflation.

3.4 Intersecting S-branes

The solutions considered so far are all vacuum solutions. Here we consider those with nonvanishing field strengths. In particular, we examine some intersecting S-brane solutions. According to the intersection rules [5, 22], we can construct an SM2-SM5 intersecting solution that leads to a 4-dimensional universe as follows:

	1	2	3	4	5	6	7	8	9	10	0
SM5	○	○	○	○	○	○					
SM2	○	○					○				

where the world-volume directions of the S-branes are indicated with a circle, and the remaining directions $(8, 9, 10, 0)$ correspond to our spacetime. We find that the solution is given by [5]

$$\begin{aligned} ds_{11}^2 &= K^{1/3} L^{2/3} \Big[\mathrm{e}^{-2\gamma_1 t} \left(-e^{6g(t)} dt^2 + \mathrm{e}^{2g(t)} d\Sigma_{3,\epsilon}^2 \right) + K^{-1} L^{-1} \mathrm{e}^{-2\gamma_1 t} (dy_1^2 + dy_2^2) + \\ &\qquad + L^{-1}\, \mathrm{e}^{\gamma_1 t} (dy_3^2 + \cdots + dy_6^2) + K^{-1}\, \mathrm{e}^{4\gamma_1 t} dy_7^2 \Big] \\ &= K^{1/3} L^{2/3} \mathrm{e}^{-2\gamma_1 t} ds_E^2 + K^{-2/3} L^{-1/3} \mathrm{e}^{-2\gamma_1 t} (dy_1^2 + dy_2^2) + \\ &\quad + K^{1/3} L^{-1/3} \mathrm{e}^{\gamma_1 t} (dy_3^2 + \cdots + dy_6^2) + K^{-2/3} L^{2/3} \mathrm{e}^{4\gamma_1 t} dy_7^2, \end{aligned} \tag{3.30}$$

where

$$K(t) = \cosh(\gamma_2\, t) \,, \qquad L(t) = \cosh(\gamma_3\, t) \,, \tag{3.31}$$

where γ_1, γ_2 and γ_3 are integration constants and $\beta = \sqrt{(18\gamma_1^2 + \gamma_2^2 + \gamma_3^2)/12}$ in $g(t)$. The metric for our universe ds_E^2 coincides with vacuum solution (3.17) with $g(t)$ given in (3.18). So unfortunately for this case we find the same behavior as in subsection 3.2, and do not get accelerated expansion. Actually it turns out that this behavior is universal to all intersecting S-brane solutions as long as we take the hyperbolic space to be the external space as we show next.

JHEP10(2003)058

3.5 General intersecting solutions

The general intersecting solution in D-dimensional supergravity coupled to a dilaton and n_A-forms is given by [5]

$$
\begin{aligned}
ds_D^2 &= \prod_A [\cosh \tilde{c}_A (t-t_A)]^{2\frac{q_A+1}{\Delta_A}} \left[e^{2c_0 t + 2c_0'} \left\{ -e^{2dg(t)} dt^2 + e^{2g(t)} d\Sigma_{d,\epsilon}^2 \right\} + \right. \\
&\qquad \left. + \sum_{\alpha=1}^{p} \prod_A [\cosh \tilde{c}_A (t-t_A)]^{-2\frac{\gamma_A^{(\alpha)}}{\Delta_A}} e^{2\tilde{c}_\alpha t + 2c_\alpha'} dx_\alpha^2 \right], \qquad (3.32)
\end{aligned}
$$

$$
\begin{aligned}
\phi &= \sum_A \frac{(D-2)\epsilon_A a_A}{\Delta_A} \ln \cosh \tilde{c}_A (t-t_A) + \tilde{c}_\phi t + c_\phi' , \\
E_A &= \sqrt{\frac{2(d-2)}{\Delta_A}} \frac{e^{\tilde{c}_A(t-t_A) - \epsilon_A a_A c_\phi'/2 + \sum_{\alpha\in q_A} c_\alpha'}}{\cosh \tilde{c}_A (t-t_A)} , \qquad \tilde{c}_A = \sum_{\alpha\in q_A} c_\alpha - \frac{1}{2} c_\phi \epsilon_A a_A , \qquad (3.33) \\
c_0 &= \sum_A \frac{q_A+1}{\Delta_A} \tilde{c}_A - \frac{\sum_{\alpha=1}^p c_\alpha}{d-1} , \qquad c_0' = - \frac{\sum_{\alpha=1}^p c_\alpha'}{d-1} , \\
\tilde{c}_\alpha &= c_\alpha - \sum_A \frac{\delta_A^{(\alpha)}}{\Delta_A} \tilde{c}_A , \qquad \tilde{c}_\phi = c_\phi + \sum_A \frac{(d-2)\epsilon_A a_A}{\Delta_A} \tilde{c}_A , \\
&\frac{1}{d-1} \left(\sum_{\alpha=1}^p c_\alpha \right)^2 + \sum_{\alpha=1}^p c_\alpha^2 + \frac{1}{2} c_\phi^2 = d(d-1)\beta^2 , \qquad (3.34)
\end{aligned}
$$

where $D = d+1+p$, and A denotes the kinds of q_A-branes, the time derivative of E_A gives the values of field strengths of the antisymmetric tenors, a_A is the parameter for the coupling of dilaton and forms, and $\epsilon_A = +1(-1)$ corresponds to electric (magnetic) fields.

Comparing eqs. (3.32) and (2.9), we find that the prefactor in front of the $(d+1)$-dimensional line element is given by

$$
\prod_A [\cosh \tilde{c}_A (t-t_A)]^{2\frac{q_A+1}{\Delta_A}} e^{2c_0 t + 2c_0'} , \qquad (3.35)
$$

and so the metric for our universe is again precisely given by (3.17). We conclude that this class of compactified theories cannot give accelerated expansion except for the critical models discussed in Subsecs. 3.1.2 and 3.3. This conclusion is valid for the solutions to the vacuum Einstein equations and also S-brane type solutions with nonvanishing field strengths.

4. Product spaces: effective action viewpoint

4.1 Dimensional reduction and effective action

In [10, subsection 2.2], and in [14, 8], pure gravity in higher dimensions is rewritten as a lower dimensional effective theory of gravity coupled to scalar fields by dimensional reduction. The advantage of using the effective theory formulation, compared with the higher dimensional gravity viewpoint, is that we can directly deal with the $(d+1)$-dimensional

JHEP10(2003)058

scale factor in the Einstein frame. Furthermore, our experience in scalar field theories will be useful in solving the equations of motion. For the reader's convenience, we reproduce here some of the formulas that will be used below.

The full spacetime is assumed to be a product space $\mathbb{R} \times \mathbb{M}_0 \times \cdots \mathbb{M}_n$. The ansatz for the metric is

$$ds^2 = \alpha^2 a^2(-d\eta^2 + d\Sigma^2_{d,\epsilon_0}) + \sum_{i=1}^{n} a_i^2 d\Sigma^2_{m_i,\epsilon_i}\,, \tag{4.1}$$

where η is the conformal time in the $(d+1)$-dimensional Einstein frame, a is the d dimensional scale factor and

$$\alpha = \prod_{i=1}^{n} \alpha_i\,, \qquad \text{with} \qquad \alpha_i = a_i^{-\frac{m_i}{d-1}}\,. \tag{4.2}$$

Derivatives of η will be denoted by a prime: $f' = df/d\eta$.

The dimensionally reduced theory of the Einstein gravity on this space to the $(d+1)$ dimensional spacetime $\mathbb{R} \times \mathbb{M}_0$ is given by $(d+1)$ dimensional gravity coupled to scalar fields ϕ_i defined by

$$a_i = \mathrm{e}^{\phi_i}\,. \tag{4.3}$$

The kinetic and potential terms for the scalar fields are

$$K = \frac{\rho+p}{2} = \sum_{i=1}^{n} \frac{m_i(m_i+d-1)}{2(d-1)a^2}\phi_i'^2 + \sum_{i>j=1}^{n} \frac{m_i m_j}{(d-1)a^2}\phi_i'\phi_j' - \epsilon_0 \frac{d-1}{2a^2}\,, \tag{4.4}$$

$$V = \frac{\rho-p}{2} = \sum_{i=1}^{n}(-\epsilon_i)\frac{m_i(m_i-1)}{2}\mathrm{e}^{-\frac{2}{d-1}\left((m_i+d-1)\phi_i + \sum_{j\neq i}^{1\leq j\leq n} m_j\phi_j\right)} - \epsilon_0\frac{(d-1)^2}{2a^2}\,. \tag{4.5}$$

The last terms in (4.4) and (4.5) are the contributions from the curvature of the d dimensional space. From (4.5) we see that effective potentials arising from gravity in higher dimensions are of exponential form. Recently, the cosmology of multiple scalar fields with a cross-coupling exponential potential was discussed in [19].

The Einstein equations in the full spacetime are equivalent to wave equations for each scalar field driven by these exponential potentials plus the Friedman equation.

4.2 Potential for $\mathbb{M}_1 \times \mathbb{M}_2 \times \mathbb{M}_3$

Let us specialize to the product space of $(d+1)$-dimensional universe and $\mathbb{M}_1 \times \mathbb{M}_2 \times \mathbb{M}_3$ with dimensions m_1, m_2, m_3. Products of one or two spaces for extra dimensions can be obtained by setting m's to zero. The ansatz of the metric for the full spacetime is

$$ds^2 = \mathrm{e}^{2\phi(x)} ds^2_{d+1} + \sum_{i=1}^{3} \mathrm{e}^{2\phi_i(x)} d\Sigma^2_{m_i,\epsilon_i}\,, \tag{4.6}$$

where we have chosen the Einstein frame by setting

$$\phi = -\frac{\sum_i m_i\phi_i}{(d-1)}\,. \tag{4.7}$$

The kinetic terms for the scalars corresponding to the radii of each internal space in the effective theory are given by (4.4). They can be diagonalized and normalized by a field redefinition

$$\begin{aligned}\psi_1 &= \sqrt{\frac{m_1(m_1+d-1)}{d-1}}\left[\phi_1 + \frac{1}{m_1+d-1}(m_2\phi_2+m_3\phi_3)\right],\\ \psi_2 &= \sqrt{\frac{m_2(m_1+m_2+d-1)}{m_1+d-1}}\left[\phi_2 + \frac{m_3}{m_1+m_2+d-1}\phi_3\right],\\ \psi_3 &= \sqrt{\frac{m_3(m_1+m_2+m_3+d-1)}{m_1+m_2+d-1}}\phi_3\,,\end{aligned} \tag{4.8}$$

with the result

$$K = \frac{1}{2}\sum_{i=1}^{3}\dot\psi_i^2 - \epsilon_0\frac{d-1}{2a^2}\,, \tag{4.9}$$

$$V = -\sum_{i=1}^{3}\epsilon_i\frac{m_i(m_i-1)}{2}e^{\sum_a M_{ia}\psi_a} - \epsilon_0\frac{(d-1)^2}{2a^2}\,, \tag{4.10}$$

where the matrix M_{ia} is given by

$$M_{ia} = \begin{pmatrix} -2\sqrt{\frac{m_1+d-1}{(d-1)m_1}} & 0 & 0 \\ -2\sqrt{\frac{m_1}{(d-1)(m_1+d-1)}} & -2\sqrt{\frac{m_1+m_2+d-1}{m_2(m_1+d-1)}} & 0 \\ -2\sqrt{\frac{m_1}{(d-1)(m_1+d-1)}} & -2\sqrt{\frac{m_2}{(m_1+d-1)(m_1+m_2+d-1)}} & -2\sqrt{\frac{m_1+m_2+m_3+d-1}{m_3(m_1+m_2+d-1)}} \end{pmatrix}. \tag{4.11}$$

To study the properties of the potential, it is more convenient to define new independent fields as

$$\begin{aligned}\varphi_1 &\equiv 2\sqrt{\frac{m_1+d-1}{m_1(d-1)}}\psi_1\,,\\ \varphi_2 &\equiv 2\sqrt{\frac{m_1}{(d-1)(m_1+d-1)}}\psi_1 + 2\sqrt{\frac{m_1+m_2+d-1}{m_2(m_1+d-1)}}\psi_2\,,\\ \varphi_3 &\equiv 2\sqrt{\frac{m_1}{(d-1)(m_1+d-1)}}\psi_1 + 2\sqrt{\frac{m_2}{(m_1+d-1)(m_1+m_2+d-1)}}\psi_2 + \\ &\quad + 2\sqrt{\frac{m_1+m_2+m_3+d-1}{m_3(m_1+m_2+d-1)}}\psi_3\,.\end{aligned} \tag{4.12}$$

The effective potential (4.5) is then

$$V = -\sum_{i=1}^{3}\epsilon_i\frac{m_i(m_i-1)}{2}\mathrm{e}^{-\varphi_i} - \epsilon_0\frac{(d-1)^2}{2a^2}\,. \tag{4.13}$$

Clearly the potential is unbounded if any one of the ϵ_i's is positive. However, if we add contributions from antisymmetric tensors, this is modified. For instance, the contribution of the 4-form field in 11-dimensional supergravity is

$$\Delta V = b^2 \exp\left[-\frac{d(m_1\varphi_1 + m_2\varphi_2 + m_3\varphi_3)}{m_1 + m_2 + m_3 + d - 1}\right], \tag{4.14}$$

where b is a constant. Now the potential is bounded from below if $\frac{dm_i}{m_1+m_2+m_3+d-1} > 1$ for the direction $\epsilon_i = +1$ (there is no requirement in the direction $\epsilon_i = -1, 0$). There will be local minimum in the direction with $\epsilon_i = +1$, but the potential minimum is always negative. This is basically the same as what is discussed in [8] for one internal space, and it seems difficult to get big expansion. In order to have big expansion, we should have local minimum at positive value, where our universe stays for a while and expands, and then decays to lower value.

Nevertheless, it is already very interesting that a potential with local minimum is obtained in the Einstein gravity coupled to gauge fields, providing a mechanism for stabilizing the size of extra dimensions. How to stabilize the size of extra dimensions is an issue no less important than obtaining inflation. This is a direction worthy of further exploration.

Another interesting research direction is to try to obtain inflation in the present picture by introducing matter or quintessence field into the solutions. This could be done, for example, by considering D-branes in the solutions. We leave this problem also to future study.

4.3 Exact solutions for exponential potentials

In this subsection we give a general discussion on solving scalar field equations with exponential potentials in flat FLRW universe. The same technique can be used to obtain solutions on hyperbolic or spherical FLRW spacetime, as we will demonstrate in section 5.

Consider scalar fields coupled to gravity in $(d+1)$-dimensional flat FLRW spacetime. The equations of motion for the scalar fields are

$$\ddot{\psi}_a + dH\dot{\psi}_a + V_a = 0\,. \tag{4.15}$$

The Friedman equation is

$$\frac{d(d-1)}{2}H^2 = K + V\,, \qquad K \equiv \frac{1}{2}\dot{\psi}_a^2\,. \tag{4.16}$$

For the potentials

$$V = \sum_i v_i \mathrm{e}^{\sum_a M_{ia}\psi_a}\,, \tag{4.17}$$

we look for solutions of the form

$$\psi_a = \alpha_a \ln t + \beta_a, \qquad H = \frac{h}{t}\,. \tag{4.18}$$

Eq. (4.15) then implies that

$$\begin{aligned} \sum_a M_{ia}\alpha_a &= -2\,, \qquad \forall i\,, \\ \left(\frac{d(d-1)}{2}h - 1\right)\alpha_a + \sum_i u_i M_{ia} &= 0\,, \qquad u_i \equiv v_i \mathrm{e}^{\sum_b M_{ib}\beta_b}\,. \end{aligned} \tag{4.19}$$

These equations are not always consistent. To be precise, if M_{ia} is invertible, then α_a can be solved. From eq. (4.16) we find

$$h = \frac{1}{d-1}\sum_a \alpha_a^2 \,. \tag{4.20}$$

If M_{ai} is invertible, we have that

$$\alpha_a = -2\sum_i M_{ai}^{-1}\,, \tag{4.21}$$

$$h = \frac{4}{d-1}\sum_{aij} M_{ai}^{-1}M_{aj}^{-1}\,. \tag{4.22}$$

There is (eternally) accelerating expansion if $h > 1$.

The results above can be easily generalized to nonstandard kinetic term

$$K = \frac{1}{2}\sum_{A,B} g_{AB}\dot{\phi}_A\dot{\phi}_B\,. \tag{4.23}$$

The potential term is still of the form (4.17)

$$V = \sum v_i e^{\sum_A M_{iA}\phi_A}\,. \tag{4.24}$$

In this case we find that the expression for h becomes

$$h = \frac{4}{d-1}\sum_{ijAB} M_{Ai}^{-1}g_{AB}M_{Bj}^{-1}\,. \tag{4.25}$$

The ansatz (4.18) allows us to find exact solutions for product spaces with more than one independent scale factors, and is a generalization of Milne spacetime discussed in section 3. In the appendix we show how to obtain asymptotic solutions for exponential potentials starting from the same ansatz (4.18).

4.4 Check of acceleration

The general solution in subsection 4.3 provides an exact solution for the spacetime in subsection 4.2. Let us check if it has any acceleration phase. According to the formula (4.21), we find from the matrix M_{ia} in (4.11) that

$$\begin{aligned}
\alpha_1 &= \sqrt{\frac{(d-1)m_1}{m_1+d-1}}\,,\\
\alpha_2 &= \frac{(d-1)\sqrt{m_2}}{\sqrt{(m_1+d-1)(m_1+m_2+d-1)}}\,,\\
\alpha_3 &= \frac{(d-1)\sqrt{m_3}}{\sqrt{(m_1+m_2+d-1)(m_1+m_2+m_3+d-1)}}\,,
\end{aligned} \tag{4.26}$$

and the Hubble parameter $H = h/t$ is determined as

$$h = \frac{1}{d-1}\alpha_a^2 = \frac{1}{2}\left[1 - \frac{d-1}{m_1+m_2+m_3+d-1}\right]. \tag{4.27}$$

This is always less than 1 for any d, in agreement with the results in subsection 3.1.1.

5. Eternally accelerating expansion on hyperbolic space

We began our study of hyperbolic universe with hyperbolic extra dimensions in subsection 3.1.2. In subsection 3.3 an exact solution was found but it did not lead to a phase of accelerated expansion. In this section we examine the critical solutions of subsection 4.3 using the dimensionally reduced scalar field theory formulation. In the next subsection we will study the perturbations theory for this critical solution.

Let the higher-dimensional geometry be $\mathbb{R} \times \mathbb{H}_d \times \mathbb{H}_m$. The metric ansatz is

$$ds^2 = \mathrm{e}^{-\frac{2m}{d-1}\phi(t)} \left(-dt^2 + a(t)^2\, ds^2_{Hd}\right) + \mathrm{e}^{2\phi(t)} ds^2_{Hm} \,. \tag{5.1}$$

The size of $\mathbb{H}_m$ corresponds to a scalar field in the effective theory on $\mathbb{R} \times \mathbb{H}_d$. Here we write the (non-trivial) components of the Einstein tensor for arbitrary ϵ_0, ϵ_1:

$$G_{00} = -\left[\frac{\lambda}{2}\dot{\phi}^2 - \epsilon_1 \frac{m(m-1)}{2}\,\mathrm{e}^{-\frac{2\lambda}{m}\phi(t)} - \frac{d(d-1)}{2}\left(H^2 + \frac{\epsilon_0}{a^2}\right)\right], \tag{5.2}$$

$$G_{xx} = -a^2\left[\frac{\lambda}{2}\dot{\phi}^2 + \epsilon_1 \frac{m(m-1)}{2}\,\mathrm{e}^{-\frac{2\lambda}{m}\phi(t)} + \frac{(d-1)(d-2)}{2}\left(H^2 + \frac{\epsilon_0}{a^2}\right) + (d-1)\frac{\ddot{a}}{a}\right], \tag{5.3}$$

$$\begin{aligned} G_{ii} = \Bigg[&\frac{\lambda}{m}\left(\ddot{\phi} + d\,H\dot{\phi}\right) - \frac{\lambda}{2}\dot{\phi}^2 - \epsilon_1 \frac{(m-1)(m-2)}{2}\,\mathrm{e}^{-\frac{2\lambda}{m}\phi(t)} - \\ &- \frac{d(d-1)}{2}\left(H^2 + \frac{\epsilon_0}{a^2}\right) - d\,\frac{\ddot{a}}{a}\Bigg]\mathrm{e}^{\frac{2\lambda}{m}\phi(t)} \,, \end{aligned} \tag{5.4}$$

where $\lambda \equiv \frac{m(m+d-1)}{(d-1)}$. The metric on $\mathbb{R} \times \mathbb{H}_d \times \mathbb{H}_m$ space takes the values $\epsilon_0 = \epsilon_1 = -1$.

Henceforth we shall take $d = 3$. Then by a change of variable

$$\phi = \sqrt{\frac{2}{m(m+2)}}\psi + \frac{1}{m+2}\ln(m(m-1))\,, \tag{5.5}$$

we can simplify the Friedman equation and the wave equation for ψ as

$$3H^2 = \frac{1}{2}\dot{\psi}^2 + \frac{1}{2}\mathrm{e}^{-c\psi} + \frac{3}{a^2}\,, \qquad c = \sqrt{\frac{2(m+2)}{m}}\,, \tag{5.6}$$

$$\ddot{\psi} + 3H\dot{\psi} - \frac{c}{2}\mathrm{e}^{-c\psi} = 0\,. \tag{5.7}$$

It is straightforward to obtain the critical solution with $\ddot{a} = 0$

$$a = \sqrt{\frac{m+2}{2}}t\,, \qquad \psi = \frac{2}{c}\ln(t) + \frac{1}{c}\ln\left(\frac{c^2}{8}\right). \tag{5.8}$$

5.1 Scalar perturbations

The exact solution with constant expansion is one of the solutions we have focused on in this paper. The main reason is that they are the solutions which critically differentiate between accelerating expansion and decelerating expansion. Any tiny perturbation should lead to interesting behavior and hopefully an accelerating phase. We now turn to possible perturbations and show how they may lead to solutions with eternally accelerating expansion.

Let us consider a small perturbation around the solution (5.8). Let

$$a = a_0 + a_1, \qquad \psi = \psi_0 + \psi_1 \,, \tag{5.9}$$

where a_0 and ψ_0 are given by (5.8). It follows that the Hubble parameter is

$$H = H_0 + H_1, \qquad H_0 = \frac{1}{t}\,, \qquad H_1 = \frac{\dot{a}_1}{a_0} - H_0\frac{a_1}{a_0}\,, \tag{5.10}$$

to the first order approximation. Expanding the equations (5.6) and (5.7) and keeping first order terms only, we get

$$6H_0H_1 = \dot{\psi}_0\dot{\psi}_1 - \frac{c}{2}\mathrm{e}^{-c\psi_0}\psi_1 - 6\frac{a_1}{a_0^3}\,, \tag{5.11}$$

$$\ddot{\psi}_1 + 3H_0\dot{\psi}_1 + 3H_1\dot{\psi}_0 + \frac{c^2}{2}\mathrm{e}^{-c\psi_0}\psi_1 = 0\,, \tag{5.12}$$

along with

$$2\dot{\psi}_0\dot{\psi}_1 + \frac{3\ddot{a}_1}{a_0} + \frac{c}{2}\,\mathrm{e}^{-c\psi_0}\psi_1 = 0\,. \tag{5.13}$$

These linear equations are easy to solve. We find the following solutions

$$a_1 = At^n\,, \qquad \psi_1 = \gamma A t^{n-1}\,, \tag{5.14}$$

where

$$\gamma = \frac{3(1-n)}{2\sqrt{m}}\,; \qquad n = \pm\sqrt{\frac{m-6}{m+2}}\,. \tag{5.15}$$

These give real solutions only if

$$m > 6\,. \tag{5.16}$$

Note that $m = 6$ or $n = 0$ is excluded because it is just a zero mode corresponding to time-shift symmetry. There are solutions with eternally accelerating expansion when $m \geq 7$, although $m = 7$ is perhaps the most interesting case. It is a very intriguing numerical coincidence that $m = 7$, which (together with our spacetime 4 dimensions) is precisely the dimension in which M-theory lives, is the minimum dimension for which this class of perturbative solutions is allowed. This coincidence suggests that the approach is worth serious consideration. The parameter A is not fixed, except that it has to be small so that the perturbative expansion is valid.

For the case $m = 7$, we have

$$n = \frac{1}{3}\,, \quad -\frac{1}{3}\,; \qquad \gamma = \frac{1}{\sqrt{7}}\,, \quad \frac{2}{\sqrt{7}}\,. \tag{5.17}$$

Examining the $n = 1/3$ solution we find that

$$a_1 = At^{1/3}\,, \qquad \ddot{a} = -\frac{2A}{9t^{5/3}}\,, \tag{5.18}$$

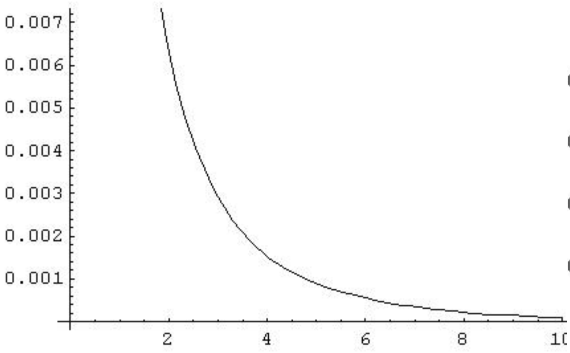

Figure 1: Acceleration $\ddot{a}$.

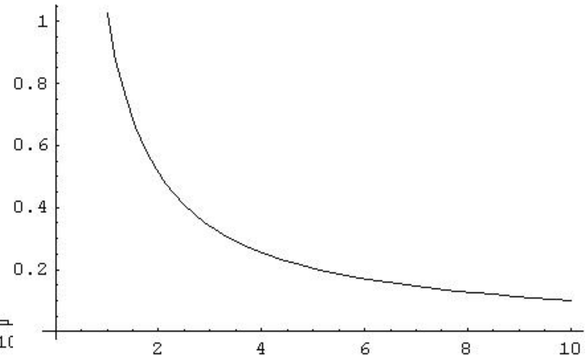

Figure 2: $\Delta a(t)$.

so this gives positive acceleration for $A < 0$. However, as time increases, a_1 grows and perturbative expansion is no longer valid. We can not claim eternally accelerating expansion for this solution without further analysis. The other solution $n = -1/3$,

$$a_1 = At^{-1/3}\,, \qquad \ddot{a} = \frac{4A}{9t^{7/3}}\,, \tag{5.19}$$

gives a positive acceleration for $A > 0$. As time increases, a_1 approaches to zero and our perturbative calculation remains valid. We find eternally accelerating expansion for this case.

Numerical solutions can be explored to verify our claim of eternally accelerating expansion without recourse to perturbation. For the initial conditions

$$a = 2.2\,, \qquad \phi = 0.18\,, \qquad \dot{\phi} = 0.2 \tag{5.20}$$

given at $t = 1$, we find that the acceleration of the scale factor a is always positive but asymptotes to zero, as shown in figure 1. In figure 2, the deviation of $a(t)$ from the critical solution (5.8),

$$\Delta a \equiv \frac{a(t) - 3t/\sqrt{2}}{3t/\sqrt{2}}\,, \tag{5.21}$$

is also shown to approach zero. This solution corresponds to $A \simeq 0.01$ at $t = 1$.

For product space compactifications, it is generally difficult to find exact solutions for the coupled Einstein equations unless the internal space is a product of flat spaces and at most one nontrivial curved space or they all are of the same type. It would be interesting to find the exact solution corresponding to the solution we have obtained here with eternally accelerating expansion and see if the inflation is further increased at early times and not just at late times.

5.2 Turning on field strengths

In this subsection we would like to investigate the effects of a non-zero form-field in a hyperbolic compactification by turning on background field strengths, and study their effects on the 4D spacetime evolution. For simplicity we consider the full spacetime to be the product of the large four-dimensions $\mathbb{R} \times \mathbb{H}_3$ and an extra hyperbolic space $\mathbb{H}_m$ of dimensions m. We will focus more on this case because we have learned that an effective gravity model where the usual $4D$ spacetime and internal spaces both are hyperbolic may increase the amount of inflation.

In order to preserve the isotropy and homogeneity of the full spacetime, the background field strength should respect the isometry. For magnetic (electric) field background, the field strength should be of the form

$$F = f(t)\,\epsilon\,, \qquad (F = f(t)\,dt \wedge \epsilon,)\,, \tag{5.22}$$

where ϵ is the volume form of either $\mathbb{H}_3$ or $\mathbb{H}_m$. Due to electric magnetic duality, we only have to consider, say, the electric and magnetic background (5.22) with ϵ being the volume form on $\mathbb{H}_3$.

5.2.1 Electric field background

Let us consider an example following [23] where a field strength is coupled to gravity. The model is the bosonic part of 11D supergravity, so we have a 3-form field A. The action is just

$$S = \int d^{11}x\sqrt{-g}\left(R - \frac{1}{2\times 4!}F_{MNPQ}F^{MNPQ}\right). \tag{5.23}$$

We take the ansatz in the Einstein conformal frame [23]

$$ds_{11}^2 = \mathrm{e}^{-7\phi}(-dt^2 + a^2 ds_3^2) + e^{2\phi}ds_7^2\,, \tag{5.24}$$

$$A_{abc} = \sqrt{g_3}\epsilon_{abc}A(t)\,, \tag{5.25}$$

where ds_3^2 (ds_7^2) is for a 3-(7-)dimensional space, and g_3 is the metric of ds_3^2. The Einstein equations are

$$G_{MN} = R_{MN} - \frac{1}{2}g_{MN}R = T_{MN}\,, \tag{5.26}$$

where

$$T_{MN} = \begin{cases} T_{\mu\nu} = -g_{\mu\nu}F^2(t)\,, & \text{for} \quad \mu,\nu = 0,1,2,3\,, \\ T_{mn} = g_{mn}F^2(t), & \text{for} \quad m,n = 4,\cdots,10\,, \end{cases} \tag{5.27}$$

where $F(t) \equiv \mathrm{e}^{14\phi}a^{-3}\dot{A}(t)/2$. The wave equation for $F(t)$ is

$$\frac{d}{dt}(\mathrm{e}^{7\phi}F(t)) = 0\,. \tag{5.28}$$

This can be immediately solved as

$$F = f\mathrm{e}^{-7\phi}\,, \tag{5.29}$$

where f is some constant.

An exact solution was given in [23] with

$$a \propto t \tag{5.30}$$

for ds_3^2 to be hyperbolic and ds_7^2 flat. This is the critical case of zero acceleration. We can hope for the existence of other solutions with acceleration, as we now show.

JHEP10(2003)058

If ds_3^2 and ds_7^2 are both taken to be hyperbolic, the 11-dimensional field equations are summarized by

$$3H^2 = \frac{63}{4}\dot{\phi}^2 + \frac{3}{a^2} + 21e^{-9\phi} + f^2 e^{-21\phi}\,, \tag{5.31}$$

$$\ddot{\phi} + 3H\dot{\phi} - 6e^{-9\phi} - \frac{2}{3}f^2 e^{-21\phi} = 0\,. \tag{5.32}$$

The terms proportional to f^2 in (5.31) and (5.32) are negligible compared to the other terms for large t. It is easy to understand why this is the case, since the f^2 term is proportional to $e^{-21\phi}$, while the other potential term is proportional to $e^{-9\phi}$. (Note that ϕ increases with time in (5.8).)

We can analyze the effect of the electric background field by treating it as a small perturbation, and repeating the same technique utilized in section 5.1, for the zeroth order solution (5.8). The only difference is that the electric field serves as a source term for the first order perturbations of the scale factor a and the scalar field ϕ. The solution is again of the form (5.14), but now with the amplitude A fixed by the source term

$$A = \frac{2^{11/6} f^2}{1215}\,, \tag{5.33}$$

and

$$n = -\frac{5}{3}\,. \tag{5.34}$$

Once again we find a perturbation which leads to acceleration for all time. As t grows larger, the magnitudes of the perturbations become smaller and the perturbative calculation remains valid. Note that, while the perturbation around the critical case can be either accelerating or decelerating when there is no background tensor field, there is only an accelerating perturbative solution when we turn on the background field. In this sense the background field assists the acceleration of the large dimensions.

5.2.2 Magnetic field background

The form field background is given by

$$F_{abc} = 2f\sqrt{g_3}\,\epsilon_{abc}\,, \tag{5.35}$$

where f is some constant, $\sqrt{g_3}$ is the (time independent) unit volume of the three space, and ϵ is the (constant) totally antisymmetrized tensor for directions a, b, $c = 1, 2, 3$, and it vanishes for other (transverse) directions. The energy momentum tensor due to the form field is[1]

$$T_{00} = -\frac{g_3}{\tilde{g}_3} g_{00} f^2\,, \qquad T_{ab} = \frac{g_3}{\tilde{g}_3} g_{ab} f^2\,, \qquad T_{mn} = -\frac{g_3}{\tilde{g}_3} g_{mn} f^2\,, \tag{5.36}$$

where $\tilde{g}_3$ is the determinant for the large spatial dimensions $\mathbb{H}_3$ including the time dependent overall scale factor.

For the ansatz

$$ds^2 = e^{-m\phi}(-dt^2 + a^2 ds_{H3}^2) + e^{2\phi} ds_{Hm}^2\,, \tag{5.37}$$

[1]The convention of the form field appearing in the action is chosen to be $-\frac{1}{2\times 3!}F_{MNP}F^{MNP}$.

in which $\tilde{g}_3 = \mathrm{e}^{-3m\phi}a^6 g_3$, the Einstein equations $G_{MN} = T_{MN}$ are equivalent to

$$\ddot{\phi} + 3H\dot{\phi} - (m-1)\,\mathrm{e}^{-(m+2)\phi} + \frac{4\mathrm{e}^{2m\phi}}{(m+2)a^6} f^2 = 0\,, \tag{5.38}$$

$$3H^2 = \frac{m(m+2)}{4}\dot{\phi}^2 + \frac{3}{a^2} + \frac{m(m-1)}{2}\,\mathrm{e}^{-(m+2)\phi} + \frac{\mathrm{e}^{2m\phi}}{a^6} f^2\,. \tag{5.39}$$

In addition to the hyperbolic curvature term, ϕ has the effective potential

$$V = \frac{(m-1)}{(m+2)}\mathrm{e}^{-(m+2)\phi} + \frac{2}{m(m+2)}\frac{\mathrm{e}^{2m\phi}}{a^6} f^2\,, \tag{5.40}$$

with a positive definite minimum for any given scale factor a, and thus one may expect that eternally accelerating expansion is possible in this case. However, just like the electric field background, the form field effect can be ignored when a gets large. Therefore the asymptotic behavior should be the same as the case without form field background.

Similar to the previous section on electric field background, we can treat the effect of the magnetic field background by perturbation theory. As an example, for $m = 7$, the solution is given by (5.14) with

$$A = -\frac{3^{17/3} f^2}{2^{19/18}}, \quad n = \frac{1}{9}\,. \tag{5.41}$$

This again gives a positive acceleration for the scale factor. In this case the first order term of the scale factor grows with time and the perturbative calculation can not be trusted for large t. But solutions with eternally accelerating expansion should exist because eventually the magnetic field can be ignored and we come back to the case in section 5.1. Comparing this case with the electric field background, we see that magnetic field background has a more significant influence on the acceleration of the universe.

6. Conclusion

In this paper we have studied the possibility of generating inflation from gravity on product spaces with and without form fields. For the only situation in which we obtained eternally accelerating expansion in subsections 5.1 and 5.2, the amount of inflation was mainly boosted by the curvature of the hyperbolic external space, when the initial condition is right. We give an explicit description of the accelerating solution as a perturbation around a critical case of constant expansion. When there is no background form field, perturbation around the critical case can be either accelerating or decelerating. A tensor field background will increase the amount of acceleration, and we find that a magnetic field in the large 3 dimensions (or equivalently the electric field in extra dimensions) has a more significant effect than the electric field in the large 3 dimensions (or magnetic field in extra dimensions). All our other examples showed inflationary phases with an e-folding number of order one.

A comment however is needed for the eternally accelerating expansion we found. The number of e-foldings during an inflationary phase is

$$\ln\frac{a(t_f)}{a(t_i)} = \int_{t_i}^{t_f} dt H(t)\,, \tag{6.1}$$

where t_i and t_f are the starting and ending time of the inflationary phase. According to this definition, a large e-folding number does not imply fast inflation. For our models with eternally accelerating expansion in subsections 5.1 and 5.2, the e-folding numbers may be large mainly because although the inflation is small it lasts forever, $t_f \to \infty$. In fact, since the curvature decreases when the space expands, the curvature of the external space becomes negligible within a short time, and the expansion of the universe is negligibly better than constant expansion. This scenario is apparently not suitable for cosmological inflation of the early universe. As for the acceleration of the present universe, we recall that, even for flat external space, the short inflationary phase due to hyperbolic extra dimensions may be used to explain the acceleration of the present universe with some fine-tuning [11]. Turning on negative curvature for the large dimensions should help, but it might not play an important role since our large spatial dimensions are known to be nearly flat.

Our study suggests that a further generalization of the known S-brane solutions would be desirable if we wish to use this model for inflation. In addition to the obvious possibility of adding positive scalar potentials by hand, or by quantum or stringy corrections, pure gravity with warped or twisted geometry has not yet been extensively studied. The possibility of introducing matter (quintessence fields) by considering D-branes or other extended objects ample in string/M-theory is another arena to study. We save these topics for future study.

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A. Asymptotic solutions for exponential potentials

In this appendix we demonstrate that the ansatz (4.18) can be used to obtain asymptotic solutions for exponential potentials.

As an example, consider the two field model with the kinetic and potential terms

$$K = \frac{1}{2}(\dot{\psi}_1^2 + \dot{\psi}_2^2), \quad V_2 = \epsilon_1 e^{c\psi_1} + \epsilon_2 e^{d\psi_1 + f\psi_2} \,. \tag{A.1}$$

Note that a more generic potential of the form

$$V = v_1 e^{a\psi_1 + b\psi_2} + v_2 e^{c\psi_1 + d\psi_2} \tag{A.2}$$

is equivalent to (A.1) by a rotation and translation of ψ_i's for $\epsilon_1 = \mathrm{sgn}(v_1), \epsilon_2 = \mathrm{sgn}(v_2)$.

The equations of motion are

$$\ddot{\psi}_1 + 3H\dot{\psi}_1 + cV_1 + dV_2 = 0\,, \tag{A.3}$$

$$\ddot{\psi}_2 + 3H\dot{\psi}_2 + fV_2 = 0\,, \tag{A.4}$$

$$3H^2 = K + V\,. \tag{A.5}$$

For large (cosmic) time t, we take the ansatz

$$\psi_1 = \alpha_1 \ln t + \beta_1, \qquad \psi_2 = \alpha_2 t^{-n} + \beta_2\,, \qquad H = \frac{h}{t}\,, \tag{A.6}$$

for $n > 0$. The idea is that the two equations of motion scale differently for large t. Since ψ_1 decays slower than ψ_2, we can have a nontrivial solution if V_2 decays faster than V_1 and drops out of (A.3), but remains significant in (A.4).

The solution is

$$\alpha_1 = -\frac{2}{c}, \quad n = \frac{2(d-c)}{c}\,, \qquad h = \frac{2}{c^2}, \quad u_1 = \frac{2(3h-1)}{c^2}\,, \qquad u_2 = \frac{1}{f} n(3h-n-1)\alpha_2\,, \tag{A.7}$$

where

$$u_1 = e^{c\beta_1}\,, \qquad u_2 = e^{d\beta_1 + f\beta_2}\,. \tag{A.8}$$

Note that α_2 is not fixed by the field equations, but the Hubble parameter is fixed.

The solution above is valid when the following conditions are met. It is obvious that $n > 0$ only if $d > c$. The remaining conditions are $\mathrm{sgn}(u_1) = \epsilon_1$ and $\mathrm{sgn}(u_2) = \epsilon_2$.

Comparing the result with (4.20), we see that the Hubble parameter here looks as if the field ψ_2 is absent. It is possible that a two-field model has eternally accelerating expansion of this kind but not the kind given in the previous section. To determine whether there is eternally accelerating solution in this section, only the sign of f is important; its magnitude is not. But for the previous section the magnitude of f is also important.

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